Homework Set 3
Physics 319
Classical Mechanics

Problem 5.13

a) To find the equilibrium position (where there is no force) set the derivative of the potential to zero

$$\frac{dU}{dr} = U_0\left(\frac{1}{R} - \lambda^2\frac{R}{r^2}\right)$$

$$\frac{dU}{dr} = 0 \text{ at } r^2 = \lambda^2 R^2 \quad r = \lambda R$$

b) If $x = r - \lambda R$ is much smaller than λR, the second order expansion of $1/(1+x)$ is needed to obtain the first significant term in the potential.

$$U(x) = U_0\left(\frac{x+\lambda R}{R} + \lambda\frac{1}{1+x/\lambda R}\right) \doteq$$

$$U_0\lambda + U_0\frac{x}{R} + U_0\lambda\left(1 - \frac{x}{\lambda R} + \frac{x^2}{\lambda^2 R^2} - \cdots\right)$$

$$= 2U_0\lambda + \frac{2U_0}{2\lambda R^2}x^2 - \cdots$$

$$k = \frac{2U_0}{\lambda R^2} \qquad \omega = \sqrt{\frac{2U_0}{m\lambda R^2}}$$

Problem 5.17

a) Suppose

$$x(t) = A_x\cos(\omega_x t)$$

$$y(t) = A_y\cos(\omega_y t - \delta)$$

and $\omega_x/\omega_y = p/q$, where p and q are the lowest integers that specify the rational number ratio. Because $q\omega_x = p\omega_y$, one defines the period of the common oscillation frequency $\tau = 2\pi/q\omega_x = 2\pi/p\omega_y$. Now after the period $T = pq\tau = 2\pi p/\omega_x = 2\pi q/\omega_y$, the motion repeats because

$$x(t+T) = A_x\cos(\omega_x t + \omega_x T) = A_x\cos(\omega_x t + 2\pi p) = A_x\cos(\omega_x t) = x(t)$$

$$y(t+T) = A_y\cos(\omega_y t + \omega_y T - \delta) = A_y\cos(\omega_y t + 2\pi q - \delta) = A_y\cos(\omega_y t - \delta) = y(t)$$

b) One way to characterize an irrational number is as a number whose decimal expansion never repeats. Suppose one approximates the frequency ratio first by its 100 decimal expansion, then its 200 decimal expansion, and so forth. By part a) the repetition period of the 100 decimal expansion is $2\pi q/\omega_y = 2\pi 10^{100}/\omega_y$, the repetition period of the 200 digit decimal expansion is $2\pi 10^{200}/\omega_y$, and so forth. If the expansion of the frequency ratio never repeats, and the repetition period gets longer the closer one gets to the actual irrational value, at the actual irrational value the pattern never repeats.

Problem 5.23

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\left[\frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2\right] \\ &= m\dot{x}\ddot{x} + kx\dot{x} = m\dot{x}\left(\ddot{x} + \omega x\right) \\ &= -b\dot{x}^2\end{aligned}$$

by equation 5.24. By the work-energy theorem, the rate that work is dissipated by the damping force is

$$\dot{W} = -F\dot{x} = b\dot{x}^2$$

Problem 5.42

$$\omega = \sqrt{\frac{g}{L}} = 0.57155 \text{ sec}^{-1}$$

If the exponential damping time is 8 hrs = 28800 sec, the *Q*-value is

$$Q = \frac{1}{2}\frac{\omega}{\beta} = (14400)(0.57155) = 8230 .$$

Problem 5.51

For the forcing function to be real, the f_n are all real. Now

$$f(t) = \sum_{n=0}^{\infty} f_n \cos(n\omega t) = \sum_{n=0}^{\infty} f_n \operatorname{Re}\left(e^{in\omega t}\right) = \operatorname{Re}\left(\sum_{n=0}^{\infty} f_n e^{in\omega t}\right) = \operatorname{Re}\left(g(t)\right)$$

For an individual n, the solution to the driven oscillator problem

$$\frac{d^2 z_n}{dt^2} + 2\beta\frac{dz_n}{dt} + \omega_0^2 z_n = \frac{f_n}{2}\left(e^{in\omega t} + e^{-in\omega t}\right)$$

is, by superposing the solutions from the individual terms on the RHS

$$\begin{aligned}z_n(t) &= \frac{f_n}{2\left(-n^2\omega^2 + 2\beta in\omega + \omega_0^2\right)} e^{in\omega t} + \frac{f_n}{2\left(-n^2\omega^2 - 2\beta in\omega + \omega_0^2\right)} e^{-in\omega t} \\ &= \operatorname{Re}\left(\frac{f_n}{\omega_0^2 - n^2\omega^2 + 2\beta in\omega} e^{in\omega t}\right)\end{aligned}$$

If $x_n(t) = \operatorname{Re}\left(f_n e^{in\omega t} / \left(\omega_0^2 - n^2\omega^2 + 2\beta in\omega\right)\right)$, summing over all the *n*s yields the required result.

Problem 6.12

The stationary condition is, from the Euler-Lagrange equation

$$L = x\sqrt{1-y'^2} \qquad \frac{d}{dx}\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y} = 0$$

By evaluating the proper derivatives, the equation for the stationary solution is

$$\frac{d}{dx}\frac{-xy'}{\sqrt{1-y'^2}} = 0 \rightarrow C^2\left(1-y'^2\right) = x^2 y'^2$$

$$C^2 = \left(C^2 + x^2\right) y'^2$$

$$y' = \frac{C}{\left(C^2 + x^2\right)^{1/2}}$$

where the integration constant in x is chosen to be C^2 for future convenience. The integral is the standard form

$$y = C\sinh^{-1}\left(x/C\right) + D$$

where D is the second integration constant.

Problem 6.16

The formula for the distance between two points on a sphere, Problem 6.1, is

$$D = R\int_{\theta_1}^{\theta_2} \sqrt{1+\sin^2\theta\phi'^2(\theta)}\,d\theta$$

(This comes from the Pythagorean Theorem applied to a small displacement on the sphere. Then $dD = \sqrt{\left(Rd\theta\right)^2 + \left(R\sin\theta d\phi\right)^2}$. This comment is not part of the solution.) D is stationary when

$$\frac{d}{d\theta}\frac{\partial L}{\partial \phi'} - \frac{\partial L}{\partial \phi} = 0 \rightarrow \frac{\partial L}{\partial \phi'} = C = \frac{\sin^2\theta\phi'}{\sqrt{1+\sin^2\theta\phi'^2(\theta)}}$$

is constant as a function of θ. Using the suggested trick, if the first point is aligned with the *z*-axis. Any path passing through the pole necessarily has $\phi' = 0$ nearby the pole and as $\sin\theta = 0$ at the pole, $C = 0$. Therefore, for the geodesic, $\phi = const$ as a function of θ. The geodesic is a longitude line at the pole, i.e., it is a great circle of the sphere.

It's not too bad to just integrate the equation for ϕ'

$$C^2\left(1+\sin^2\theta\phi'^2\right)=\sin^4\theta\phi'^2 \to \phi' = C/\left(\sin^4\theta - C^2\sin^2\theta\right)^{1/2}$$

$$\phi = \int \frac{C}{\left(\sin^4\theta - C^2\sin^2\theta\right)^{1/2}}d\theta = \int \frac{\csc^2\theta}{\left(\left(1/C^2-1\right)-\cot^2\theta\right)^{1/2}}d\theta$$

$$\phi = -\int \frac{1}{\left(\left(1/C^2-1\right)-y^2\right)^{1/2}}dy \to -\sin\left(\phi-\phi_0\right) = \frac{\cot\theta}{\left(1/C^2-1\right)^{1/2}}$$

$$-\sin\theta\sin\phi\cos\phi_0 + \sin\theta\cos\phi\sin\phi_0 - \frac{\cos\theta}{\left(1/C^2-1\right)^{1/2}} = 0$$

These coordinates are on the plane passing through the center of the sphere.

Problem 6.18

The distance function in polar coordinates is $D = \int_{\theta_1}^{\theta_2} \sqrt{1+r^2\left(d\theta/dr\right)^2}\,dr$ or $D = \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(dr/d\theta\right)^2}\,d\theta$ because $dD = \sqrt{r^2 d\theta^2 + dr^2}$. It turns out using expression 1 is "the easy" way, and expression 2 is a bit more involved. Using the Euler-Lagrange equation on expression 1 yields

$$\frac{d}{dr}\frac{r^2 d\theta/dr}{\sqrt{1+r^2\left(d\theta/dr\right)^2}} = 0$$

$$r^4\left(\frac{d\theta}{dr}\right)^2 = C^2\left(1+r^2\left(\frac{d\theta}{dr}\right)^2\right)$$

$$\frac{d\theta}{dr} = \frac{C}{r\left(r^2-C^2\right)^{1/2}} \to d\theta = \frac{Cdr}{r\left(r^2-C^2\right)^{1/2}}$$

The integral is a standard one that may be solved by the substitution
$r = C/\cos\hat{\theta},\ dr = C\sin\hat{\theta}/\cos^2\hat{\theta}$

$$d\theta = \frac{C^2\sin\hat{\theta}d\hat{\theta}}{\cos^2\hat{\theta}}\frac{\cos\hat{\theta}}{C}\frac{\cos\hat{\theta}}{\left(C^2-C^2\cos^2\hat{\theta}\right)^{1/2}} = d\hat{\theta}$$

$$\theta-\theta_0 = \hat{\theta} = \cos^{-1}\left(\frac{C}{r}\right) \to r\left(\theta\right) = \frac{C}{\cos\left(\theta-\theta_0\right)}$$

This expression is the polar equation for a line: C is identified as the distance of closest approach of the line to the origin and θ_0 is the angle the line makes with the y-axis, positive being angle in counterclockwise orientation.

Using expression 2 for the distance, the Euler-Lagrange equation is

$$\int_{r_1,\theta_1}^{r_2,\theta_2} \sqrt{r^2+\left(dr/d\theta\right)^2}\,d\theta$$

$$\frac{d}{d\theta}\frac{dr/d\theta}{\sqrt{r^2+\left(dr/d\theta\right)^2}} = \frac{r}{\sqrt{r^2+\left(dr/d\theta\right)^2}}$$

Because (this trick discussed in Problem 6.20)

$$\frac{d}{d\theta}\sqrt{r^2+\left(dr/d\theta\right)^2} = \frac{r\left(dr/d\theta\right)}{\sqrt{r^2+\left(dr/d\theta\right)^2}} + r''\frac{d}{d\theta}\frac{\partial L}{\partial r'}$$

$$= \left(dr/d\theta\right)\frac{d}{d\theta}\frac{\partial L}{\partial r'} + r''\frac{d}{d\theta}\frac{\partial L}{\partial r'} = \frac{d}{d\theta}\left[r'\frac{\partial L}{\partial r'}\right],$$

integrating in θ yields

$$\sqrt{r^2+\left(dr/d\theta\right)^2} = \frac{\left(dr/d\theta\right)^2}{\sqrt{r^2+\left(dr/d\theta\right)^2}} + C$$

$$\sqrt{r^2+\left(dr/d\theta\right)^2} - \frac{\left(dr/d\theta\right)^2}{\sqrt{r^2+\left(dr/d\theta\right)^2}} = C$$

$$r^2+\left(dr/d\theta\right)^2 - 2\left(dr/d\theta\right)^2 + \frac{\left(dr/d\theta\right)^4}{r^2+\left(dr/d\theta\right)^2} = C^2$$

$$r^4 = C^2\left(r^2+\left(dr/d\theta\right)^2\right) \to \frac{dr}{d\theta} = r\left(r^2/C^2-1\right)^{1/2}$$

$$\cos^{-1}\frac{C}{r} = \theta+\theta_0$$

Problem 6.26

The integral to extremize is

$$S = \int_{u_1}^{u_2} f\left[x(u), y(u), x'(u), y'(u), u\right]du$$

As is the book's argument, assume a small deviation from the stationary solution

$$x = x(u) + \alpha\delta x(u)$$

$$y = y(u) + \beta\delta y(u)$$

$$x' = x'(u) + \alpha\delta x'(u)$$

$$y' = y'(u) + \beta\delta y'(u)$$

For the solution to be stationary

$$\frac{\partial S}{\partial \alpha} = \int_{u_1}^{u_2} f\left[x(u)+\alpha\delta x(u), y(u)+\beta\delta y(u), x'(u)+\alpha\delta x'(u), y'(u)+\beta\delta y'(u), u\right]du = 0$$

$$\frac{\partial S}{\partial \beta} = \int_{u_1}^{u_2} f\left[x(u)+\alpha\delta x(u), y(u)+\beta\delta y(u), x'(u)+\alpha\delta x'(u), y'(u)+\beta\delta y'(u), u\right]du = 0$$

$$\int_{u_1}^{u_2}\left[\frac{\partial f}{\partial x}\delta x(u)+\frac{\partial f}{\partial x'}\frac{d\delta x}{du}(u)\right]du = 0$$

$$\int_{u_1}^{u_2}\left[\frac{\partial f}{\partial x}\delta x(u)+\frac{\partial f}{\partial x'}\frac{d\delta x}{du}(u)\right]du = 0$$

$$\int_{u_1}^{u_2}\left[\frac{\partial f}{\partial x}-\frac{d}{du}\frac{\partial f}{\partial x'}\right]\delta x(u)\,du = 0$$

$$\int_{u_1}^{u_2}\left[\frac{\partial f}{\partial y}-\frac{d}{du}\frac{\partial f}{\partial y'}\right]\delta y(u)\,du = 0$$

Because these final two integrals must vanish for all variations $\delta x(u)$ and $\delta y(u)$, the Euler-Lagrange equations follow

$$\frac{d}{du}\frac{\partial f}{\partial x'}-\frac{\partial f}{\partial x} = 0$$
$$\frac{d}{du}\frac{\partial f}{\partial y'}-\frac{\partial f}{\partial y} = 0$$

Problem 6.27

The distance function to extremize is

$$D = \int\sqrt{(dx/du)^2+(dy/du)^2+(dz/du)^2}\,du$$

Applying the Euler-Lagrange equations for the three coordinates in turn

$$\frac{d}{du}\frac{dx/du}{\sqrt{(dx/du)^2+(dy/du)^2+(dz/du)^2}} = 0$$

$$\frac{d}{du}\frac{dy/du}{\sqrt{(dx/du)^2+(dy/du)^2+(dz/du)^2}} = 0$$

$$\frac{d}{du}\frac{dz/du}{\sqrt{(dx/du)^2+(dy/du)^2+(dz/du)^2}} = 0$$

$(dx/du, dy/du, dz/du)$ is a constant vector as a function of u.

$$\therefore \vec{x}(u) = \vec{x}_0 + (d\vec{x}/du)u \quad \text{where } d\vec{x}/du \equiv (\vec{x}_1-\vec{x}_0)/\Delta u$$